7.20:

g prodotto scalare su $\mathbb{R}^3$ con segnatura $(2,1,0)$

Che segnatura può avere $g|_W$ con $W \subseteq \mathbb{R}^3$ piano?

Si può trovare W con segnatura $(2,0,0)$, $(1,1,0)$, $(1,0,1)$

NON " " " " " " $(0,2,0)$, $(0,1,1)$, $(0,0,2)$

Svolgimento:

Sylvester: $\exists\, \mathcal{B}$ ortonormale t.c. $[g]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = S$

$\mathcal{B} = \{v_1, v_2, v_3\}$

$W = \operatorname{Span}(v_1, v_2)$ $\quad [g|_W]_{v_1,v_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ def + $(2,0,0)$

$W = \operatorname{Span}(v_2, v_3)$ $\quad [g|_W]_{v_2,v_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $(1,1,0)$

Cerco un $W = \mathrm{Span}(v, w)$ t.c. $[g|_W]_{v,w} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$(1,0,1)$

II $\to$ prendo v ortogonale a w

I $\to$ prendo w isotropo

$$\boxed{\begin{aligned} g(v,w) &= 0 \\ g(w,w) &= 0 \end{aligned}} \qquad g(v,v) = 1$$

$$[w]_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad g(w,w) = (x_1\ x_2\ x_3)\, S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$0 = g(w,w) = x_1^2 + x_2^2 - x_3^2 \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = S$$

Es: $[w]_B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ è isotropo

Cerco v ortogonale a w che sia indipendente da w

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (x_1 \ x_2 \ x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{[w]_B} = 0$$

$$x_1 - x_3 = 0$$

$$\text{Es: } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = [v]_B$$

$$W = \mathrm{Span}(v, w)$$

$$[g|_W]_{v,w} = \begin{pmatrix} g(v,v) & g(v,w) \\ g(v,w) & g(w,w) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$g(v,v) = g(v_2, v_2) = 1$$

$$v = v_2$$

$$w = v_1 + v_3$$

$$\boxed{S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$$

$$\mathrm{sgn} = (2, 1, 0)$$

$$S' \equiv S \quad \text{congruente} \Rightarrow$$

(Sylvester) $\Rightarrow \exists\ \mathcal{C} = \{w_1, w_2, w_3\}$ t.c.

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = S}$$

$$[g]_{\mathcal{C}} = S'$$

Prendo $W = \mathrm{Span}(w_1, w_2)$

$\nexists\ W \subseteq \mathbb{R}^3$ t.c. $g|_W$ sia : $\cancel{(0,2,0)}$, $\cancel{(0,1,1)}$, $(0,0,2)$

$W \subseteq V$
$$\boxed{\begin{array}{l} i_+(g|_W) \leq i_+(g) \\ i_-(g|_W) \leq i_-(g) \end{array}}$$
segue dalla definizione

NON È VERO CHE $i_0(g|_W) \leq i_0(g)$!

Per assurdo: W t.c. $g|_W$ abbia segnatura $(0,1,1)$

$W = \mathrm{Span}(v_1, v_2)$. Completo a base v_1, v_2, v_3 di $\mathbb{R}^3$

ortogonali

$$S = [g]_{v_1,v_2,v_3} = \begin{pmatrix} -1 & 0 & a \\ 0 & 0 & b \\ a & b & c \end{pmatrix} \quad a,b,c \in \mathbb{R} \text{ che non conosco}$$

Mostro che S non ha la segnatura di g $(2,1,0)$

i_dispari

$$\det S = -1 \cdot (-b^2) = b^2 \geq 0 \quad \text{assurdo}$$

Ultimo caso: stesso ragionamento $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & c \end{pmatrix} = S'$

$$\det S' = 0 \Rightarrow \text{assurdo}$$

II metodo: $(2,1,0)$ g $\quad \nexists W$ con $g|_W$ $\cancel{(0,1,1)}$ (semidef.) $\cancel{(0,0,2)}$ nullo

p. ass. $\exists W$ con $g|_W$ semidef.

$\exists\, U \subseteq \mathbb{R}^3$ def + $\quad \dim(\underset{\substack{2 \\ >0}}{U} \cap \underset{\substack{2 \\ \leq 0}}{W}) \geq 1$

piano

$\Rightarrow \exists\, v \neq 0,\ v \in U \cap W$

$v \in U \Rightarrow g(v,v) > 0$

$v \in W \Rightarrow g(v,v) \leq 0$

7.16:

$$S = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & \alpha \\ 1 & \alpha & \alpha^2 \end{pmatrix} \quad \text{segnatura?} \qquad \alpha \in \mathbb{R}$$

$$d_1 = 1 \qquad d_2 = -4 \qquad d_3 = \det S = -2\left(2\alpha^2 - \alpha\right) - \alpha(\alpha - 2) =$$

$$= -4\alpha^2 + 2\alpha - \alpha^2 + 2\alpha = \alpha(-5\alpha + 4)$$

$d_3 = 0$ per $\alpha = 0 \vee \alpha = \frac{4}{5}$ $\qquad d_3 > 0$ per $\alpha \in (0, 4/5)$

$d_3 < 0$ per $\alpha < 0 \vee \alpha > 4/5$

Per $\alpha \in (0, 4/5)$: $\quad \overset{+}{1}, \overset{+}{d_1}^{\,1}, \overset{-}{d_2}^{\,-4}, \overset{+}{d_3}$ $\qquad i_+ = 1,\ i_- = 2 \Rightarrow \boxed{(1,2,0)}$

Per $\alpha < 0 \vee \alpha > 4/5$ $\quad + \ + \ - \ -$ $\qquad i_+ = 2,\ i_- = 1 \Rightarrow \boxed{(2,1,0)}$

$\alpha = 0$? $\qquad \alpha = \frac{4}{5}$?

$$\det < 0 \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad i_0 = 1 \qquad (i_+, i_-) = (2,0), (1,1) \text{ o } (0,2)$$

2×2 con $\det < 0 \Rightarrow (1,1) \Rightarrow$ $i_+ \geq 1$, $i_- \geq 1$

$$\alpha = 4/5 \qquad S = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 4/5 \\ 1 & 4/5 & 16/25 \end{pmatrix} \qquad rk = 2$$

come prima

Se $\alpha = 0 \vee \alpha = 4/5$ $\qquad \boxed{(1,1,1)}$

det < -1

$$S = \begin{pmatrix} \boxed{\alpha} & \alpha+1 & \alpha+2 \\ \alpha+1 & \alpha+2 & \alpha+1 \\ \alpha+2 & \alpha+1 & \alpha \end{pmatrix}$$

$d_1 = \alpha$

$$d_2 = \alpha(\alpha+2) - (\alpha+1)^2 = -1$$

$d_3 =$

$$\det S = \det \begin{pmatrix} \alpha & \alpha+1 & \alpha+2 \\ 1 & 1 & -1 \\ 2 & 0 & -2 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 2\alpha+2 \\ 1 & 1 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

$\mathrm{II} \to \mathrm{II} - \mathrm{I}$

$\mathrm{III} \to \mathrm{III} - \mathrm{I}$

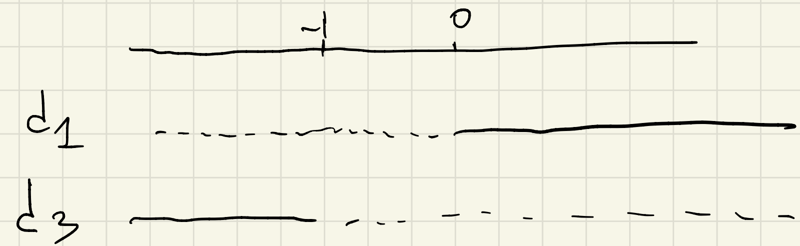
$\mathrm{I} \to \mathrm{I} - \alpha\,\mathrm{II}$

$$= (-1) \det \begin{pmatrix} 1 & 2\alpha+2 \\ 0 & -2 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 1 & 2\alpha+2 \\ 1 & -1 \end{pmatrix}$$

$$= 2 + 2(-1 - 2\alpha - 2) = -4\alpha - 4$$

$$d_3 = \det S = \begin{cases} < 0 & \text{se } \alpha > -1 \\ 0 & \text{se } \alpha = -1 \\ > 0 & \text{se } \alpha < -1 \end{cases}$$

$$\begin{array}{cccc} 1, & d_1, & d_2, & d_3 \\ & \alpha & -1 & -4\alpha - 4 \end{array}$$



- $\alpha < -1$: $\quad 1, d_1, d_2, d_3 \quad$ (+ − − +) $\quad (1, 2, 0)$
- $-1 < \alpha < 0$: $\quad$ (+ − − −) $\quad (2, 1, 0)$
- $\alpha > 0$: $\quad$ (+ + − −) $\quad (2, 1, 0)$

Mancano $\alpha = -1,\ 0$

$$\alpha = -1: \quad \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad i_0 = 1 \implies (1,1,1)$$

det<0

$i_+ \geq 1 \qquad i_- \geq 1$

det<0

$$\alpha = 0: \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad i_0 = 0$$

$i_+ \geq 1 \qquad i_- \geq 1 \implies (2,1,0) \vee (1,2,0)$

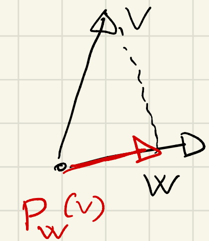
$\det S < 0$

$\det S < 0$

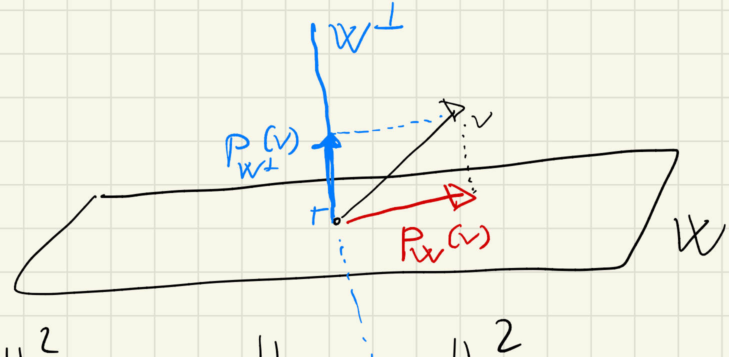
$\alpha > -1 \qquad \det S < 0 \to (2,1,0)$

$\to \cancel{(0,2,0)}$ det < 0

Proiezioni ortogonali: $\quad V$ sp. vett. def $+$



$$w \neq 0 \qquad P_w(v) = \frac{\langle v,w\rangle}{\langle w,w\rangle}\, w$$



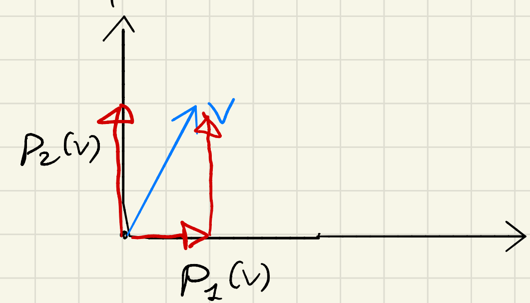
Teo (Pitagora in n dimensioni):

V sp. vett. def. $+$ $\quad \mathcal{B} = \{v_1, \dots, v_n\}$ ortogonale

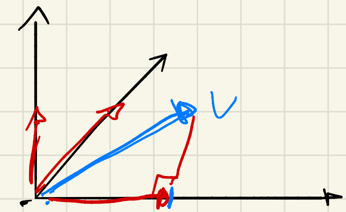
$\forall v \in V$:

$$\|v\|^2 = \|P_{v_1}(v)\|^2 + \|P_{v_2}(v)\|^2 + \dots + \|P_{v_n}(v)\|^2$$

$$\|v\|^2 = \|P_1(v)\|^2 + \|P_2(v)\|^2$$



dim: $v = P_{v_1}(v) + \dots\dots + P_{v_n}(v)$



$$\|v\|^2 = \langle v, v \rangle =$$

$$= \langle P_{v_1}(v) + \dots + P_{v_n}(v), P_{v_1}(v) + \dots + P_{v_n}(v) \rangle$$

$$= \sum_{i,j=1}^{n} \underbrace{\langle P_{v_i}(v), P_{v_j}(v) \rangle} = \sum_{i=1}^{n} \langle P_{v_i}(v), P_{v_i}(v) \rangle =$$

se $i \neq j$ $\langle v_i, v_j \rangle = 0$

Quindi $\langle P_{v_i}(v), P_{v_j}(v) \rangle = 0$

$$= \sum_{i=1}^{n} \|P_{v_i}(v)\|^2$$

Ortogonalizzazione di Gram-Schmidt

Metodo che trasforma base $v_1, \text{---}, v_n$ di V in base ortogonale

$$\Downarrow$$

$w_1, \dots, w_n$ ortogonale

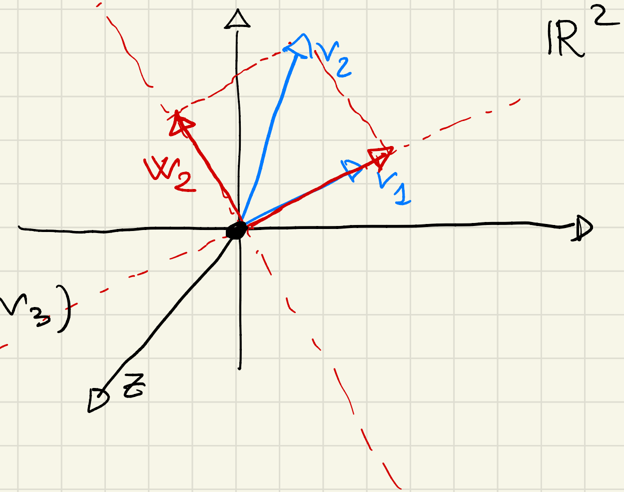
$$w_1 = v_1$$

$$w_2 = v_2 - P_{w_1}(v_2)$$

$$w_3 = v_3 - P_{w_1}(v_3) - P_{w_2}(v_3)$$

$$\vdots$$

$$w_n = v_n - P_{w_1}(v_n) - \dots \quad \dots - P_{w_{n-1}}(v_n)$$



Esempio: $\mathbb{R}^2$ $\quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \langle v_1, v_2 \rangle = 5 \neq 0$

$$w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$w_2 = v_2 - P_{w_1}(v_2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - 5/2 \\ 3 - 5/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

Nota: si può cambiare w_2 riscalandolo

$$\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \times 2 \to \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ numeri semplici

norma = 1

$$\begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right\}$$

$$w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$w_2 = v_2 - \underbrace{\frac{\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{= P_{w_1}(v_2)}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = w_2$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = w_3 \qquad w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Eventualmente si può normalizzare e ottenere una base ortonormale:

$$w_i \longrightarrow \overline{w_i} = \frac{w_i}{\|w_i\|}$$

$$\overline{w_1} = \frac{w_1}{\|w_1\|} \quad \frac{w_1}{\sqrt{2}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$$

$$\overline{w_2} = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{6}} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Prop: Se $v_1, \dots, v_k \in V$ non nulli e ortogonali, allora sono indipendenti.

dim: Supponiamo che $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$

$$\Rightarrow \quad \forall i \quad \langle \lambda_1 v_1 + \dots + \lambda_k v_k , v_i \rangle = 0$$

$$\underbrace{\lambda_1 \underbrace{\langle v_1, v_i \rangle}_{} + \dots + \lambda_k \underbrace{\langle v_k, v_i \rangle}_{}}_{} = 0$$

$$\lambda_i \underbrace{\langle v_i, v_i \rangle}_{\neq 0} = 0 \Rightarrow \lambda_i = 0 \qquad \square$$

Cor: Ogni $v_1, \dots, v_k$ ortogonali possono sempre essere completati a base ortogonale $v_1, \dots, v_k, \dots, v_n$

Es: Se v_1, v_2 sono ortogonali in $\mathbb{R}^3$, posso prendere

$v_3 = v_1 \times v_2$ e ottengo v_1, v_2, v_3 base ortogonale di $\mathbb{R}^3$

dim: $v_1, \dots, v_k$ ortogonali $\Rightarrow$ indip. $\Rightarrow$ completo a base $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ $\Rightarrow$ usare Gram-Schmidt per sistemare $v_{k+1}, \dots, v_n$

Es: $\mathbb{R}^3$ $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ $v_3 = v_1 \times v_2 = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$

$\mathbb{R}^4$ $w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $w_2 = v_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix}$ ortogonali

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

G.S.: $$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{10} \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \frac{1}{4} - \frac{2}{5} \\ \vdots \\ \\ \end{pmatrix}$$

$$w_4 = v_4 - \frac{\langle v_4, w_1 \rangle}{\langle w_1, w_1 \rangle} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\langle v_4, w_2 \rangle}{\langle w_2, w_2 \rangle} \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix} - \frac{\langle v_4, w_3 \rangle}{\langle w_3, w_3 \rangle} \begin{pmatrix} ? \end{pmatrix}$$

Prop Disuguaglianza di Bessel :

V sp. vett. con prod. scalare def +

$v_1, \dots, v_k \in V$ ortogonali

- Se questi sono una base: $\forall v \in V$,

$$\|v\|^2 = \|P_{v_1}(v)\|^2 + \dots + \|P_{v_n}(v)\|^2$$

Volta scorsa: $\|P_{v_1}(v)\| = \left\| \frac{\langle v_1, v\rangle}{\langle v_1, v_1\rangle} v_1 \right\| = \frac{|\langle v_1, v\rangle|}{\|v_1\|}$

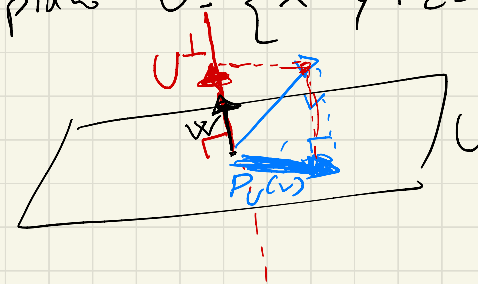
$$\|v\|^2 = \sum_{i=1}^{n} \|P_{v_i}(v)\|^2 = \sum_{i=1}^{n} \frac{\langle v_i, v\rangle^2}{\|v_i\|^2}$$

<u>Bessel</u>: · In generale, se non sono una base ho $\geq$ al posto di $=$

$$\boxed{\|v\|^2 \geq \sum_{i=1}^{k} \frac{\langle v_i, v\rangle^2}{\|v_i\|^2}}$$

<u>Es</u>: Scrivi la proiezione ortogonale sul piano $U = \{x - y + z = 0\}$ rispetto alla base canonica

$$P_U : \mathbb{R}^3 \to \mathbb{R}^3$$
$$v \longmapsto P_U(v)$$



$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boxed{P_U(v) = v - P_{U^\perp}(v)}$$

$$0 \neq w \in U^\perp$$

$$w = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$P_U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - P_w \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{x - y + z}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x - \frac{x-y+z}{3} \\ y + \frac{x-y+z}{3} \\ z - \frac{x-y+z}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x + \frac{1}{3}y - \frac{1}{3}z \\ \frac{1}{3}x + \frac{2}{3}y + \frac{1}{3}z \\ -\frac{1}{3}x + \frac{1}{3}y + \frac{2}{3}z \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A matrice che rappresenta P_U rispetto alla canonica

$$P_U = L_A$$

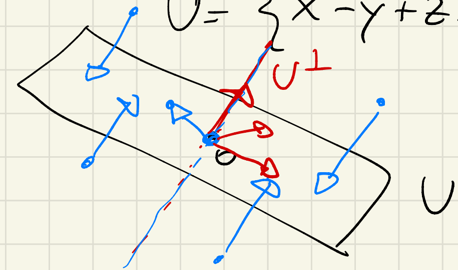
Esercizi: 8.1 → 8.11

VEN: 8.4 8.8 solo proiez.

Verifiche:

$$\frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = A$$

$U = \{x - y + z = 0\}$



$$\operatorname{Ker} p_U = \operatorname{Ker} A = U^\perp = V_0$$

$$\operatorname{Im} p_U = \operatorname{Im} A = U = V_1 = \operatorname{Fix}(p_U)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V = U \oplus U^\perp = V_0 \oplus V_1$$

$$\operatorname{Ker} A = \operatorname{Span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \operatorname{Im} A = U$$